

Critical Independent Sets of a Graph

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Abstract

Let G be a simple graph with vertex set $V(G)$. A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\text{Ind}(G)$ we mean the family of all independent sets of G .

The number $d(X) = |X| - |N(X)|$ is the *difference* of $X \subseteq V(G)$, and a set $A \in \text{Ind}(G)$ is *critical* if $d(A) = \max\{d(I) : I \in \text{Ind}(G)\}$ [26].

Let us recall the following definitions:

$$\text{core}(G) = \bigcap \{S : S \text{ is a maximum independent set}\} \quad [10],$$

$$\text{corona}(G) = \bigcup \{S : S \text{ is a maximum independent set}\} \quad [2],$$

$$\text{ker}(G) = \bigcap \{S : S \text{ is a critical independent set}\} \quad [12],$$

$$\text{diadem}(G) = \bigcup \{S : S \text{ is a critical independent set}\}.$$

In this paper we present various structural properties of $\text{ker}(G)$, in relation with $\text{core}(G)$, $\text{corona}(G)$, and $\text{diadem}(G)$.

Keywords: independent set, critical set, ker , core , corona , diadem , matching

1 Introduction

Throughout this paper G is a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $X \subseteq V(G)$, then $G[X]$ is the subgraph of G induced by X . By $G - W$ we mean either the subgraph $G[V(G) - W]$, if $W \subseteq V(G)$, or the subgraph obtained by deleting the edge set W , for $W \subseteq E(G)$. In either case, we use $G - w$, whenever $W = \{w\}$. If $A, B \subseteq V(G)$, then (A, B) stands for the set $\{ab : a \in A, b \in B, ab \in E(G)\}$.

The *neighborhood* $N(v)$ of a vertex $v \in V(G)$ is the set $\{w : w \in V(G) \text{ and } vw \in E(G)\}$, while the *closed neighborhood* $N[v]$ of $v \in V(G)$ is the set $N(v) \cup \{v\}$; in order to avoid ambiguity, we use also $N_G(v)$ instead of $N(v)$.

The *neighborhood* $N(A)$ of $A \subseteq V(G)$ is $\{v \in V(G) : N(v) \cap A \neq \emptyset\}$, and $N[A] = N(A) \cup A$. We may also use $N_G(A)$ and $N_G[A]$, when referring to neighborhoods in a graph G .

A set $S \subseteq V(G)$ is *independent* if no two vertices from S are adjacent, and by $\text{Ind}(G)$ we mean the family of all the independent sets of G . An independent set of maximum size is a *maximum independent set* of G , and the *independence number* $\alpha(G)$ of G is $\max\{|S| : S \in \text{Ind}(G)\}$. Let $\Omega(G)$ denote the family of all maximum independent sets, and let

$$\begin{aligned} \text{core}(G) &= \bigcap \{S : S \in \Omega(G)\} \text{ [10], and} \\ \text{corona}(G) &= \cup \{S : S \in \Omega(G)\} \text{ [2].} \end{aligned}$$

Clearly, $N(\text{core}(G)) \subseteq V(G) - \text{corona}(G)$, and there are graphs with $N(\text{core}(G)) \neq V(G) - \text{corona}(G)$ (for an example, see Figure 1). The problem of whether $\text{core}(G) \neq \emptyset$ is **NP**-hard [2].

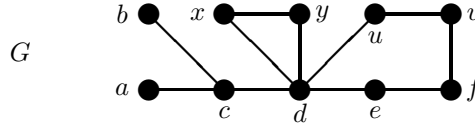


Figure 1: $\text{core}(G) = \{a, b\}$ and $V(G) - \text{corona}(G) = N(\text{core}(G)) \cup \{d\} = \{c, d\}$.

A *matching* is a set M of pairwise non-incident edges of G . If $A \subseteq V(G)$, then $M(A)$ is the set of all the vertices matched by M with vertices belonging to A . A matching of maximum cardinality, denoted $\mu(G)$, is a *maximum matching*.

For $X \subseteq V(G)$, the number $|X| - |N(X)|$ is the *difference* of X , denoted $d(X)$. The *critical difference* $d(G)$ is $\max\{d(X) : X \subseteq V(G)\}$. The number $\max\{d(I) : I \in \text{Ind}(G)\}$ is the *critical independence difference* of G , denoted $id(G)$. Clearly, $d(G) \geq id(G)$. It was shown in [26] that $d(G) = id(G)$ holds for every graph G . If A is an independent set in G with $d(X) = id(G)$, then A is a *critical independent set* [26]. All pendant vertices not belonging to K_2 components are included in every inclusion maximal critical independent set.

For example, let $X = \{v_1, v_2, v_3, v_4\}$ and $I = \{v_1, v_2, v_3, v_6, v_7\}$ in the graph G of Figure 2. Note that X is a critical set, since $N(X) = \{v_3, v_4, v_5\}$ and $d(X) = 1 = d(G)$, while I is a critical independent set, because $d(I) = 1 = id(G)$. Other critical sets are $\{v_1, v_2\}$, $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_3, v_4, v_6, v_7\}$.

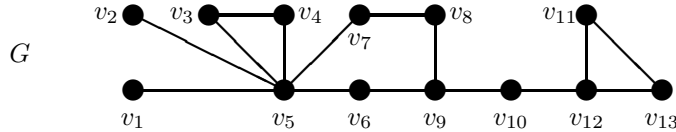


Figure 2: $\text{core}(G) = \{v_1, v_2, v_6, v_{10}\}$ is a critical set.

It is known that finding a maximum independent set is an **NP**-hard problem [7]. Zhang proved that a critical independent set can be found in polynomial time [26]. A simpler algorithm, reducing the critical independent set problem to computing a maximum independent set in a bipartite graph is given in [1].

Theorem 1.1 [3] *Each critical independent set can be enlarged to a maximum independent set.*

Theorem 1.1 led to an efficient way of approximating $\alpha(G)$ [25]. Moreover, it has been shown that a critical independent set of maximum cardinality can be computed in polynomial time [8]. Recently, a parallel algorithm computing the critical independence number was developed [5].

Recall that if $\alpha(G) + \mu(G) = |V(G)|$, then G is a *König-Egerváry graph* [6, 24]. As a well-known example, each bipartite graph is a König-Egerváry graph as well.

Theorem 1.2 [11] *If G is a König-Egerváry graph, M is a maximum matching of G , and $S \in \Omega(G)$, then:*

- (i) M matches $V(G) - S$ into S , and $N(\text{core}(G))$ into $\text{core}(G)$;
- (ii) $N(\text{core}(G)) = \cap \{V(G) - S : S \in \Omega(G)\}$, i.e., $N(\text{core}(G)) = V(G) - \text{corona}(G)$.

The *deficiency* $\text{def}(G)$ is the number of non-saturated vertices relative to a maximum matching, i.e., $\text{def}(G) = |V(G)| - 2\mu(G)$ [19]. A proof of a conjecture of Graffiti.pc [4] yields a new characterization of König-Egerváry graphs: these are exactly the graphs, where there exists a critical maximum independent set [9]. In [13] it is proved the following.

Theorem 1.3 [13] *For a König-Egerváry graph G the following equalities hold*

$$d(G) = |\text{core}(G)| - |N(\text{core}(G))| = \alpha(G) - \mu(G) = \text{def}(G).$$

Using this finding, we have strengthened the characterization from [9].

Theorem 1.4 [13] *G is a König-Egerváry graph if and only if each of its maximum independent sets is critical.*

For a graph G , let denote

$$\begin{aligned} \ker(G) &= \bigcap \{S : S \text{ is a critical independent set}\} \quad [12], \text{ and} \\ \text{diadem}(G) &= \bigcup \{S : S \text{ is a critical independent set}\}. \end{aligned}$$

In this paper we present several properties of $\ker(G)$, in relation with $\text{core}(G)$, $\text{corona}(G)$, and $\text{diadem}(G)$.

2 Preliminaries

Let G be the graph from Figure 2; the sets $X = \{v_1, v_2, v_3\}$, $Y = \{v_1, v_2, v_4\}$ are critical independent, and the sets $X \cap Y$, $X \cup Y$ are also critical, but only $X \cap Y$ is also independent. In addition, one can easily see that $\ker(G)$ is a minimal critical independent set of G . These properties of critical sets and $\ker(G)$ are true even in general.

Theorem 2.1 [12] *For a graph G , the following assertions are true:*

- (i) *the function d is supermodular, i.e., $d(A \cup B) + d(A \cap B) \geq d(A) + d(B)$ for every $A, B \subseteq V(G)$;*
- (ii) *if A and B are critical in G , then $A \cup B$ and $A \cap B$ are critical as well;*
- (iii) *G has a unique minimal independent critical set, namely, $\ker(G)$.*

As a consequence, we have the following.

Corollary 2.2 *For every graph G , $\text{diadem}(G)$ is a critical set.*

For instance, the graph G from Figure 2 has $\text{diadem}(G) = \{v_1, v_2, v_3, v_4, v_6, v_7, v_{10}\}$, which is critical, but not independent.

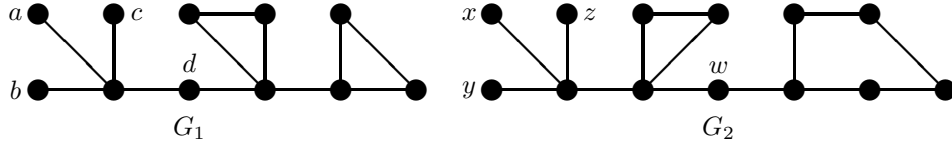


Figure 3: Both G_1 and G_2 are not König-Egerváry graphs.

The graph G from Figure 1 has $d(G) = 1$ and $d(\text{corona}(G)) = 0$, which means that $\text{corona}(G)$ is not a critical set. Notice that G is not a König-Egerváry graph. Combining Theorems 1.4 and 2.1(ii), we deduce the following.

Corollary 2.3 *If G is a König-Egerváry graph, then both $\text{core}(G)$ and $\text{corona}(G)$ are critical sets.*

Let consider the graphs G_1 and G_2 from Figure 3: $\text{core}(G_1) = \{a, b, c, d\}$ and it is a critical set, while $\text{core}(G_2) = \{x, y, z, w\}$ and it is not critical.

Theorem 2.4 *If $\text{core}(G)$ is a critical set, then*

$$\text{core}(G) \subseteq \bigcap \{A : A \text{ is an inclusion maximal critical independent set}\}.$$

Proof. Let A be an arbitrary inclusion maximal critical independent set. According to Theorem 1.1, there is some $S \in \Omega(G)$, such that $A \subseteq S$. Since $\text{core}(G) \subseteq S$, it follows that $A \cup \text{core}(G) \subseteq S$, and hence $A \cup \text{core}(G)$ is independent. By Theorem 2.1, we get that $A \cup \text{core}(G)$ is a critical independent set. Since $A \subseteq A \cup \text{core}(G)$ and A is an inclusion maximal critical independent set, it follows that $\text{core}(G) \subseteq A$, for every such set A , and this completes the proof. ■

Remark 2.5 *By Theorem 1.1 the following inclusion holds for every graph G .*

$$\text{corona}(G) \supseteq \bigcup \{A : A \text{ is an inclusion maximal critical independent set}\}.$$

3 Structural properties of $\ker(G)$

Deleting a vertex from a graph may change its critical difference. For instance, $d(G - v_1) = d(G) - 1$, $d(G - v_{13}) = d(G)$, while $d(G - v_3) = d(G) + 1$, where G is the graph of Figure 2.

Proposition 3.1 [16] *For a vertex v in a graph G , the following assertions hold:*

- (i) $d(G - v) = d(G) - 1$ if and only if $v \in \ker(G)$;
- (ii) if $v \in \ker(G)$, then $\ker(G - v) \subseteq \ker(G) - \{v\}$.

Note that $\ker(G - v)$ may differ from $\ker(G) - \{v\}$. For example, $\ker(K_{3,2})$ is equal to the partite set of size 3, but $\ker(K_{3,2} - v) = \emptyset$ whenever v is in that set. Also, if $G = C_4$, then $\ker(G) - \{v\} = \emptyset - \{v\} = \emptyset$, while $\ker(G - v) = N_G(v)$ for every $v \in V(G)$.

Theorem 3.2 [8] *There is a matching from $N(S)$ into S for every critical independent set S .*

In the graph G of Figure 2, let $S = \{v_1, v_2, v_3\}$. By Theorem 3.2, there is a matching from $N(S)$ into $S = \{v_1, v_2, v_3\}$, for instance, $M = \{v_2v_5, v_3v_4\}$, since S is critical independent. On the other hand, there is no matching from $N(S)$ into $S - v_3$.

Theorem 3.3 [16] *For a critical independent set A in a graph G , the following statements are equivalent:*

- (i) $A = \ker(G)$;
- (ii) there is no set $B \subseteq N(A)$, $B \neq \emptyset$ such that $|N(B) \cap A| = |B|$;
- (iii) for each $v \in A$ there exists a matching from $N(A)$ into $A - v$.

The graphs G_1 and G_2 in Figure 4 satisfy $\ker(G_1) = \text{core}(G_1)$, $\ker(G_2) = \{x, y, z\} \subset \text{core}(G_2)$, and both $\text{core}(G_1)$ and $\text{core}(G_2)$ are critical sets of maximum size. The graph G_3 in Figure 4 has $\ker(G_3) = \{u, v\}$, the set $\{t, u, v\}$ as a critical independent set of maximum size, while $\text{core}(G_3) = \{t, u, v, w\}$ is not a critical set.

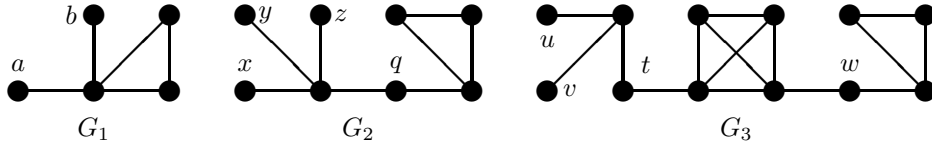


Figure 4: $\text{core}(G_1) = \{a, b\}$, $\text{core}(G_2) = \{q, x, y, z\}$, $\text{core}(G_3) = \{t, u, v, w\}$.

An independent set S is *inclusion minimal with $d(S) > 0$* if no proper subset of S has positive difference. For example, in Figure 4 one can see that $\ker(G_1)$ is an inclusion minimal independent set with positive difference, while for the graph G_2 the sets $\{x, y\}$, $\{x, z\}$, $\{y, z\}$ are inclusion minimal independent with positive difference, and $\ker(G_2) = \{x, y\} \cup \{x, z\} \cup \{y, z\}$.

Theorem 3.4 [16] *If $\ker(G) \neq \emptyset$, then*

$$\begin{aligned} \ker(G) &= \bigcup \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) = 1\} \\ &= \bigcup \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0\}. \end{aligned}$$

In a graph G , the union of all minimum cardinality independent sets S with $d(S) > 0$ may be a proper subset of $\ker(G)$. For example, consider the graph G in Figure 5, where $\{x, y\} \subset \ker(G) = \{x, y, u, v, w\}$.

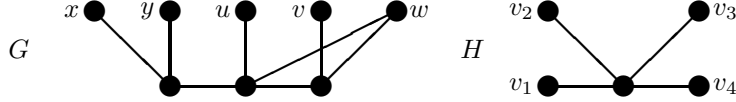


Figure 5: Both $S_1 = \{x, y\}$ and $S_2 = \{u, v, w\}$ are inclusion minimal independent sets satisfying $d(S) > 0$.

Actually, all inclusion minimal independent sets S with $d(S) > 0$ are of the same difference.

Proposition 3.5 [16] *If S_0 is an inclusion minimal independent set with $d(S_0) > 0$, then $d(S_0) = 1$. In other words,*

$$\begin{aligned} \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) > 0\} &= \\ = \{S_0 : S_0 \text{ is an inclusion minimal independent set with } d(S_0) = 1\}. \end{aligned}$$

The converse of Proposition 3.5 is not true. For instance, $S = \{x, y, u\}$ is independent in the graph G of Figure 5 and $d(S) = 1$, but S is not minimal with this property.

Proposition 3.6 [16] *$\min \{|S_0| : d(S_0) > 0, S_0 \in \text{Ind}(G)\} \leq |\ker(G)| - d(G) + 1$ is true for every graph G .*

4 Relationships between $\ker(G)$ and $\text{core}(G)$

Let us consider again the graph G_2 from Figure 3: $\text{core}(G_2) = \{x, y, z, w\}$ and it is not critical, but $\ker(G_2) = \{x, y, z\} \subseteq \text{core}(G_2)$. Clearly, the same inclusion holds for G_1 , whose $\text{core}(G_1)$ is a critical set.

Theorem 4.1 [12] *For every graph G , $\ker(G) \subseteq \text{core}(G)$.*

Let I_c be a maximum critical independent set of G , and $X = I_c \cup N(I_c)$. In [23] it is proved that $\text{core}(G[X]) \subseteq \text{core}(G)$. Moreover, in [12], we showed that the chain of relationships $\ker(G) = \ker(G[X]) \subseteq \text{core}(G[X]) \subseteq \text{core}(G)$ holds for every graph G . Theorem 4.1 allows an alternative proof of the following inequality due to Lorentzen.

Corollary 4.2 [18, 22, 12] *The inequality $d(G) \geq \alpha(G) - \mu(G)$ holds for every graph.*

Following Ore [20], [21], the number $\delta(X) = d(X) = |X| - |N(X)|$ is the *deficiency* of X , where $X \subseteq A$ or $X \subseteq B$ and $G = (A, B, E)$ is a bipartite graph. Let

$$\delta_0(A) = \max\{\delta(X) : X \subseteq A\}, \quad \delta_0(B) = \max\{\delta(Y) : Y \subseteq B\}.$$

A subset $X \subseteq A$ having $\delta(X) = \delta_0(A)$ is *A-critical*, while $Y \subseteq B$ having $\delta(Y) = \delta_0(B)$ is *B-critical*. For a bipartite graph $G = (A, B, E)$ let us denote $\ker_A(G) = \cap\{S : S \text{ is } A\text{-critical}\}$ and $\text{diadem}_A(G) = \cup\{S : S \text{ is } A\text{-critical}\}$. Similarly, $\ker_B(G) = \cap\{S : S \text{ is } B\text{-critical}\}$ and $\text{diadem}_B(G) = \cup\{S : S \text{ is } B\text{-critical}\}$.

It is convenient to define $d(\emptyset) = \delta(\emptyset) = 0$.

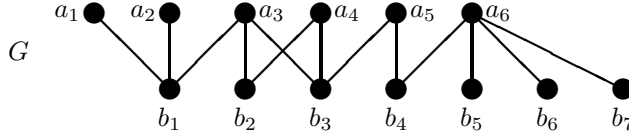


Figure 6: G is a bipartite graph without perfect matchings.

For instance, the graph $G = (A, B, E)$ from Figure 6 has: $X = \{a_1, a_2, a_3, a_4\}$ as an *A-critical* set, $\ker_A(G) = \{a_1, a_2\}$, $\text{diadem}_A(G) = \{a_i : i = 1, \dots, 5\}$ and $\delta_0(A) = 1$, while $Y = \{b_i : i = 4, 5, 6, 7\}$ is a *B-critical* set, $\ker_B(G) = \{b_4, b_5, b_6\}$, $\text{diadem}_B(G) = \{b_i : i = 2, \dots, 7\}$ and $\delta_0(B) = 2$.

As expected, there is a close relationship between critical independent sets and *A-critical* or *B-critical* sets.

Theorem 4.3 [14] *Let $G = (A, B, E)$ be a bipartite graph. Then the following assertions are true:*

- (i) $d(G) = \delta_0(A) + \delta_0(B)$;
- (ii) $\alpha(G) = |A| + \delta_0(B) = |B| + \delta_0(A) = \mu(G) + \delta_0(A) + \delta_0(B) = \mu(G) + d(G)$;
- (iii) *if X is an A-critical set and Y is a B-critical set, then $X \cup Y$ is a critical set;*
- (iv) *if Z is a critical independent set, then $Z \cap A$ is an A-critical set and $Z \cap B$ is a B-critical set;*
- (v) *if X is either an A-critical set or a B-critical set, then there is a matching from $N(X)$ into X .*

The following lemma will be used further to give an alternative proof for the assertion that $\ker(G) = \text{core}(G)$ holds for every bipartite graph G .

Lemma 4.4 *If $G = (A, B, E)$ is a bipartite graph with a perfect matching, say M , $S \in \Omega(G)$, $X \in \text{Ind}(G)$, $X \subseteq V(G) - S$, and $G[X \cup M(X)]$ is connected, then*

$$X^1 = X \cup M((N(X) \cap S) - M(X))$$

is an independent set, and $G[X^1 \cup M(X^1)]$ is connected.

Proof. Let us show that the set $M((N(X) \cap S) - M(X))$ is independent. Suppose, to the contrary, that there exist $v_1, v_2 \in M((N(X) \cap S) - M(X))$ such that $v_1 v_2 \in E(G)$. Hence $M(v_1), M(v_2) \in (N(X) \cap S) - M(X)$.

If $M(v_1)$ and $M(v_2)$ have a common neighbor $w \in X$, then $\{v_1, v_2, M(v_2), w, M(v_1)\}$ spans C_5 , which is forbidden for bipartite graphs.

Otherwise, let $w_1, w_2 \in X$ be neighbors of $M(v_1)$ and $M(v_2)$, respectively. Since $G[X \cup M(X)]$ is connected, there is a path with even number of edges connecting w_1 and w_2 . Together with $\{w_1, M(v_1), v_1, v_2, M(v_2), w_2\}$ this path produces a cycle of odd length in contradiction with the hypothesis on G being a bipartite graph.

To complete the proof of independence of the set

$$X^1 = X \cup M((N(X) \cap S) - M(X))$$

it is enough to demonstrate that there are no edges connecting vertices of X and $M((N(X) \cap S) - M(X))$.

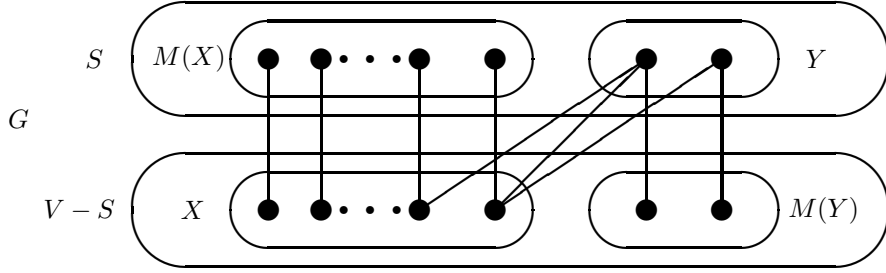


Figure 7: $S \in \Omega(G)$, $Y = (N(X) \cap S) - M(X)$ and $X^1 = X \cup M(Y)$.

Assume, to the contrary, that there is $vw \in E$, such that $v \in M((N(X) \cap S) - M(X))$ and $w \in X$. Since $M(v) \in (N(X) \cap S) - M(X)$ and $G[X \cup M(X)]$ is connected, it follows that there exists a path with an odd number of edges connecting $M(v)$ to w . This path together with the edges vw and $vM(v)$ produces cycle of odd length, in contradiction with the bipartiteness of G .

Finally, since $G[X \cup M(X)]$ is connected, $G[X^1 \cup M(X^1)]$ is connected as well, by definitions of set functions N and M . ■

Theorem 4.1 claims that $\ker(G) \subseteq \text{core}(G)$ for every graph.

Theorem 4.5 [14] *If G is a bipartite graph, then $\ker(G) = \text{core}(G)$.*

Alternative Proof. The assertions are clearly true, whenever $\text{core}(G) = \emptyset$, i.e., for G having a perfect matching. Assume that $\text{core}(G) \neq \emptyset$.

Let $S \in \Omega(G)$ and M be a maximum matching. By Theorem 1.2(i), M matches $V(G) - S$ into S , and $N(\text{core}(G))$ into $\text{core}(G)$.

According to Theorem 3.3(ii), it is sufficient to show that there is no set $Z \subseteq N(\text{core}(G))$, $Z \neq \emptyset$, such that $|N(Z) \cap \text{core}(G)| = |Z|$.

Suppose, to the contrary, that there exists a non-empty set $Z \subseteq N(\text{core}(G))$ such that $|N(Z) \cap \text{core}(G)| = |Z|$. Let Z_0 be a minimal non-empty subset of $N(\text{core}(G))$ enjoying this equality.

Clearly, $H = G[Z_0 \cup M(Z_0)]$ is bipartite, because it is a subgraph of a bipartite graph. Moreover, the restriction of M on H is a perfect matching.

Claim 1. Z_0 is independent.

Since H is a bipartite graph with a perfect matching it has two maximum independent sets at least. Hence there exists $W \in \Omega(H)$ different from $M(Z_0)$. Thus $W \cap Z_0 \neq \emptyset$. Therefore, $N(W \cap Z_0) \cap \text{core}(G) = M(W \cap Z_0)$. Consequently,

$$|N(W \cap Z_0) \cap \text{core}(G)| = |M(W \cap Z_0)| = |W \cap Z_0|.$$

Finally, $W \cap Z_0 = Z_0$, because Z_0 has been chosen as a minimal subset of $N(\text{core}(G))$ such that $|N(Z_0) \cap \text{core}(G)| = |Z_0|$. Since $|Z_0| = \alpha(H) = |W|$ we conclude with $W = Z_0$, which means, in particular, that Z_0 is independent.

Claim 2. H is a connected graph.

Otherwise, for any connected component of H , say \tilde{H} , the set $V(\tilde{H}) \cap Z_0$ contradicts the minimality property of Z_0 .

Claim 3. $Z_0 \cup (\text{core}(G) - M(Z_0))$ is independent.

By Claim 1 Z_0 is independent. The equality $|N(Z_0) \cap \text{core}(G)| = |Z_0|$ implies $N(Z_0) \cap \text{core}(G) = M(Z_0)$, which means that there are no edges connecting Z_0 and $\text{core}(G) - M(Z_0)$. Consequently, $Z_0 \cup (\text{core}(G) - M(Z_0))$ is independent.

Claim 4. $Z_0 \cup (\text{core}(G) - M(Z_0))$ is included in a maximum independent set.

Let $Z_i = M((N(Z_{i-1}) \cap S) - M(Z_{i-1}))$, $1 \leq i < \infty$. By Lemma 4.4 all the sets $Z^i = \bigcup_{0 \leq j \leq i} Z_j$, $1 \leq i < \infty$ are independent. Define

$$Z^\infty = \bigcup_{0 \leq i \leq \infty} Z_i,$$

which is, actually, the largest set in the sequence $\{Z^i, 1 \leq i < \infty\}$.

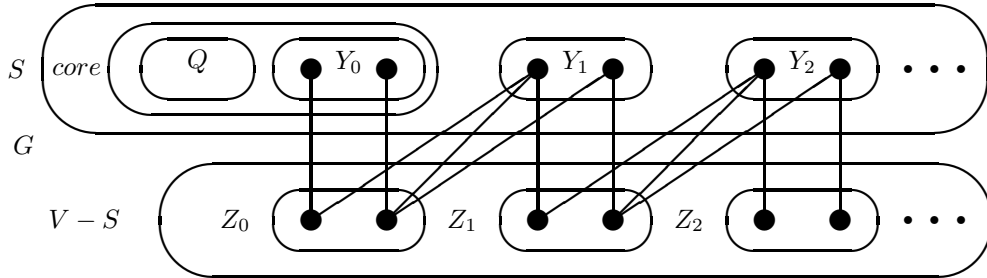


Figure 8: $S \in \Omega(G)$, $Q = \text{core}(G) - M(Z_0)$, $Y_0 = M(Z_0)$, $Y_1 = (N(Z_0) - M(Z_0)) \cap S$, $Y_2 = \dots$, and $Z_i = M(Y_i)$, $i = 1, 2, \dots$.

The inclusion

$$Z_0 \cup (\text{core}(G) - M(Z_0)) \subseteq (S - M(Z^\infty)) \cup Z^\infty$$

is justified by the definition of Z^∞ .

Since $|M(Z^\infty)| = |Z^\infty|$ we obtain $|(S - M(Z^\infty)) \cup Z^\infty| = |S|$. According to the definition of Z^∞ the set

$$(N(Z^\infty) \cap S) - M(Z^\infty)$$

is empty. In other words, the set $(S - M(Z^\infty)) \cup Z^\infty$ is independent. Therefore, we arrive at

$$(S - M(Z^\infty)) \cup Z^\infty \in \Omega(G).$$

Consequently, $(S - M(Z^\infty)) \cup Z^\infty$ is a desired enlargement of $Z_0 \cup (\text{core}(G) - M(Z_0))$.

Claim 5. $\text{core}(G) \cap ((S - M(Z^\infty)) \cup Z^\infty) = \text{core}(G) - M(Z_0)$.

The only part of $(S - M(Z^\infty)) \cup Z^\infty$ that interacts with $\text{core}(G)$ is the subset

$$Z_0 \cup (\text{core}(G) - M(Z_0)).$$

Hence we obtain

$$\begin{aligned} \text{core}(G) \cap ((S - M(Z^\infty)) \cup Z^\infty) &= \\ &= \text{core}(G) \cap (Z_0 \cup (\text{core}(G) - M(Z_0))) = \text{core}(G) - M(Z_0). \end{aligned}$$

Since Z_0 is non-empty, by Claim 5 we arrive at the following contradiction

$$\text{core}(G) \not\subseteq (S - M(Z^\infty)) \cup Z^\infty \in \Omega(G).$$

Finally, we conclude with the fact there is no set $Z \subseteq N(\text{core}(G))$, $Z \neq \emptyset$ such that $|N(Z) \cap \text{core}(G)| = |Z|$, which, by Theorem 3.3, means that $\text{core}(G)$ and $\ker(G)$ coincide. ■

Notice that there are non-bipartite graphs enjoying the equality $\ker(G) = \text{core}(G)$; e.g., the graphs from Figure 9, where only G_1 is a König-Egerváry graph.

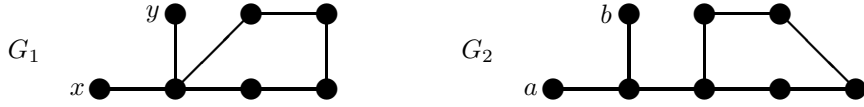


Figure 9: $\text{core}(G_1) = \ker(G_1) = \{x, y\}$ and $\text{core}(G_2) = \ker(G_2) = \{a, b\}$.

There is a non-bipartite König-Egerváry graph G , such that $\ker(G) \neq \text{core}(G)$. For instance, the graph G_1 from Figure 10 has $\ker(G_1) = \{x, y\}$, while $\text{core}(G_1) = \{x, y, u, v\}$. The graph G_2 from Figure 10 has $\ker(G_2) = \emptyset$, while $\text{core}(G_2) = \{w\}$.

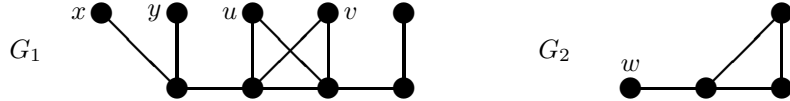


Figure 10: Both G_1 and G_2 are König-Egerváry graphs. Only G_2 has a perfect matching.

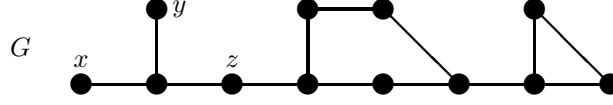


Figure 11: G is not a König-Egerváry graph, and $\text{core}(G) = \{x, y, z\}$.

5 $\ker(G)$ and $\text{diadem}(G)$ in König-Egerváry graphs

There is a non-König-Egerváry graph G with $V(G) = N(\text{core}(G)) \cup \text{corona}(G)$; e.g., the graph G from Figure 11.

Theorem 5.1 *If G is a König-Egerváry graph, then*

- (i) $|\text{corona}(G)| + |\text{core}(G)| = 2\alpha(G)$;
- (ii) $\text{diadem}(G) = \text{corona}(G)$, while $\text{diadem}(G) \subseteq \text{corona}(G)$ is true for every graph;
- (iii) $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$.

Proof. (i) Using Theorems 1.2(ii) and 1.3, we infer that

$$\begin{aligned} |\text{corona}(G)| + |\text{core}(G)| &= |\text{corona}(G)| + |N(\text{core}(G))| + |\text{core}(G)| - |N(\text{core}(G))| = \\ &= |V(G)| + d(G) = \alpha(G) + \mu(G) + d(G) = 2\alpha(G). \end{aligned}$$

as claimed.

(ii) Every $S \in \Omega(G)$ is a critical set, by Theorem 1.4. Hence we deduce that $\text{corona}(G) \subseteq \text{diadem}(G)$. On the other hand, for every graph each critical independent set is included in a maximum independent set, according to Theorem 1.1. Thus, we infer that $\text{diadem}(G) \subseteq \text{corona}(G)$. Consequently, the equality $\text{diadem}(G) = \text{corona}(G)$ holds.

(iii) It follows by combining parts (i), (ii) and Theorem 4.1. ■

Notice that the graph from Figure 11 has $|\text{corona}(G)| + |\text{core}(G)| = 13 > 12 = 2\alpha(G)$.

For a König-Egerváry graph with $|\ker(G)| + |\text{diadem}(G)| < 2\alpha(G)$ see Figure 10. Figure 11 shows that it is possible for a graph to have $\text{diadem}(G) \subsetneq \text{corona}(G)$ and $\ker(G) \subsetneq \text{core}(G)$.

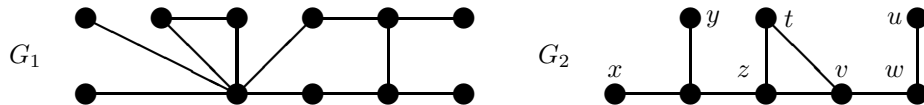


Figure 12: G_1 is a non-bipartite König-Egerváry graph, such that $\ker(G_1) = \text{core}(G_1)$ and $\text{diadem}(G_1) = \text{corona}(G_1)$; G_2 is a non-König-Egerváry graph, such that $\ker(G_2) = \text{core}(G_2) = \{x, y\}$; $\text{diadem}(G_2) \cup \{z, t, v, w\} = \text{corona}(G_2)$.

The combination of $\text{diadem}(G) \subsetneq \text{corona}(G)$ and $\ker(G) = \text{core}(G)$ is realized in Figure 12.

Proposition 5.2 *Let $G = (A, B, E)$ be a bipartite graph.*

- (i) [21] *If $X = \ker_A(G)$ and Y is a B -critical set, then $X \cap N(Y) = N(X) \cap Y = \emptyset$;*
- (ii) [20] $\ker_A(G) \cap N(\ker_B(G)) = N(\ker_A(G)) \cap \ker_B(G) = \emptyset$.

Now we are ready to describe both \ker and diadem of a bipartite graph in terms of its bipartition.

Theorem 5.3 *Let $G = (A, B, E)$ be a bipartite graph. Then the following assertions are true:*

- (i) $\ker_A(G) \cup \ker_B(G) = \ker(G)$;
- (ii) $|\ker(G)| + |\text{diadem}(G)| = 2\alpha(G)$;
- (iii) $|\ker_A(G)| + |\text{diadem}_B(G)| = |\ker_B(G)| + |\text{diadem}_A(G)| = \alpha(G)$;
- (iv) $\text{diadem}_A(G) \cup \text{diadem}_B(G) = \text{diadem}(G)$.

Proof. (i) By Theorem 4.3(iii), $\ker_A(G) \cup \ker_B(G)$ is critical in G . Moreover, the set $\ker_A(G) \cup \ker_B(G)$ is independent in accordance with Proposition 5.2(ii). Assume that $\ker_A(G) \cup \ker_B(G)$ is not minimal. Hence the unique minimal d -critical set of G , say Z , is a proper subset of $\ker_A(G) \cup \ker_B(G)$, by Theorem 2.1(iii). According to Theorem 4.3(iv), $Z_A = Z \cap A$ is an A -critical set, which implies $\ker_A(G) \subseteq Z_A$, and similarly, $\ker_B(G) \subseteq Z_B$. Consequently, we get that $\ker_A(G) \cup \ker_B(G) \subseteq Z$, in contradiction with the fact that $\ker_A(G) \cup \ker_B(G) \neq Z \subset \ker_A(G) \cup \ker_B(G)$.

(ii), (iii), (iv) By Proposition 5.2(i), we have

$$|\ker_A(G)| - \delta_0(A) + |\text{diadem}_B(G)| = |N(\ker_A(G))| + |\text{diadem}_B(G)| \leq |B|.$$

Hence, according to Theorem 4.3(ii), it follows that

$$|\ker_A(G)| + |\text{diadem}_B(G)| \leq |B| + \delta_0(A) = \alpha(G).$$

Changing the roles of A and B , we obtain

$$|\ker_B(G)| + |\text{diadem}_A(G)| \leq \alpha(G).$$

By Theorem 4.3(iv), $\text{diadem}(G) \cap A$ is A -critical and $\text{diadem}(G) \cap B$ is B -critical. Hence $\text{diadem}(G) \cap A \subseteq \text{diadem}_A(G)$ and $\text{diadem}(G) \cap B \subseteq \text{diadem}_B(G)$. It implies both the inclusion $\text{diadem}(G) \subseteq \text{diadem}_A(G) \cup \text{diadem}_B(G)$, and the inequality

$$|\text{diadem}(G)| \leq |\text{diadem}_A(G)| + |\text{diadem}_B(G)|.$$

Combining Theorem 4.5, Theorem 5.1(i),(ii), and part (i) with the above inequalities, we deduce

$$\begin{aligned} 2\alpha(G) &\geq |\ker_A(G)| + |\ker_B(G)| + |\text{diadem}_A(G)| + |\text{diadem}_B(G)| \geq \\ &\geq |\ker(G)| + |\text{diadem}(G)| = |\text{core}(G)| + |\text{corona}(G)| = 2\alpha(G). \end{aligned}$$

Consequently, we infer that

$$\begin{aligned} |\text{diadem}_A(G)| + |\text{diadem}_B(G)| &= |\text{diadem}(G)|, \\ |\ker(G)| + |\text{diadem}(G)| &= 2\alpha(G), \\ |\ker_A(G)| + |\text{diadem}_B(G)| &= |\ker_B(G)| + |\text{diadem}_A(G)| = \alpha(G). \end{aligned}$$

Since $\text{diadem}(G) \subseteq \text{diadem}_A(G) \cup \text{diadem}_B(G)$ and $\text{diadem}_A(G) \cap \text{diadem}_B(G) = \emptyset$, we finally obtain that

$$\text{diadem}_A(G) \cup \text{diadem}_B(G) = \text{diadem}(G),$$

as claimed. ■

6 Conclusions

In this paper we focus on interconnections between \ker , core , diadem , and corona . In [15] we showed that $2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|$ is true for every graph, while the equality holds whenever G is a König-Egerváry graph, by Theorem 5.1(i).

According to Theorem 4.1, $\ker(G) \subseteq \text{core}(G)$ for every graph. On the other hand, Theorem 1.1 implies the inclusion $\text{diadem}(G) \subseteq \text{corona}(G)$. Hence

$$|\ker(G)| + |\text{diadem}(G)| \leq |\text{core}(G)| + |\text{corona}(G)|$$

for each graph G . These remarks together with Theorem 5.1(iii) motivate the following.

Conjecture 6.1 $|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G)$ is true for every graph G .

When it is proved one can conclude that the following inequalities:

$$|\ker(G)| + |\text{diadem}(G)| \leq 2\alpha(G) \leq |\text{core}(G)| + |\text{corona}(G)|$$

hold for every graph G .

By Corollary 2.3, $\text{core}(G)$ is critical for every König-Egerváry graph. It justifies the following.

Problem 6.2 Characterize graphs such that $\text{core}(G)$ is a critical set.

Theorem 4.5 claims that the sets $\ker(G)$ and $\text{core}(G)$ coincide for bipartite graphs. On the other hand, there are examples showing that this equality holds even for some non-König-Egerváry graphs (see Figure 9). We propose the following.

Problem 6.3 Characterize graphs with $\ker(G) = \text{core}(G)$.

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